CONVERGENCE RATES IN THE CENTRAL LIMIT THEOREM FOR THE SUMS OF A RANDOM NUMBER OF INDEPENDENT AND IDENTICALLY DISTRIBUTED RANDOM VARIABLES

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1. Introduction and Results.

Let $\{X_n, n\geq 1\}$ be a sequence of independent and identically distributed random variables with $EX_1=0$, $EX_1^2=1$, and distribution function F. Let $\{N_n, n\geq 1\}$ be a sequence of positive integer-valued random variables independent of $\{X_n, n\geq 1\}$. The distribution function of a random variable N_n is determined by the values $p_k(n)=P(N_n=k)$ $k\geq 1$, where $p_k(n)$ are functions of n such that for all n, $p_k(n)\geq 0$ and $\sum_{k=1}^{\infty}p_k(n)=1$. Put

$$a_n=EN_n,\ a_0=0,\ M_n^2=\sigma^2N_n \text{ (assumed to be finite for all } n).$$

We shall assume that $a_0<a_1<a_2\dots$, and $\lim_{n\to\infty} a_n=\infty$.

Further on let

$$S_n=\sum_{j=1}^{n}X_j,\ S_{N_n}=\sum_{j=1}^{N_n}X_j,\ G(x)=P(|X_1|\leqq x),\ F_n(x)=P(S_{N_n}<x\sqrt{N_n}),$$

and denote by $\Phi(x)$ the standard normal distribution. By C we shall denote positive constants which are in general different.

The purpose of this paper is to give an extension of Rychlik and Szynal [9] for the sums of a random number of independent and identically distributed random variables.

The obtained results are extensions of some theorems given in [4] [5] and [9].

We shall show the following theorems.

Theorem 1. *The following inequality holds,*

$$\sup_x |F_n(x)-\Phi(x)| \leq C[a_n^{-1/2}\int_{|u|\leq\sqrt{a_n}} |u|^3 dF(u) + \int_{|u|>\sqrt{a_n}} u^2 dF(u) + M_n a_n^{-2}]$$

Theorem 2. *For all x, the following inequality holds,*

$$|F_n(x)-\Phi(x)| \leq C[(1+|x|)^{-3}\int_0^{(1+|x|)\sqrt{a_n}}\int_{|u|>v} u^2 dF(u)dv + (1+|x|)^{-2}M_n a_n^{-2}]$$

Theorem 3. *For all x, the following inequality holds,*

$$|F_n(x)-\Phi(x)| \leq C(1+|x|^2)^{-1}[a_n^{-1/2}\int_0^{\sqrt{a_n}} u^3 dG(u) + \int_{\sqrt{a_n}}^{\infty} u^2 dG(u) + M_n a_n^{-2}].$$

Heyde [5] has shown the inequality in the case of non random number of Theorem 3. Some applications of Theorem 3 are given in following corollaries which extend Theorem 4 and Theorem 5 of Rychlik and Szynal [9]. They have shown the case of $p=\infty$ in following corollaries.

Corallary 1. *Let $\{X_n, n\geq 1\}$ be a sequence of independent and identically distributed random variables such that $EX_1=0$, $EX_1^2=1$ and $E|X_1|^{2+\delta}<\infty$, where $0<\delta<1$. If there exist positive constants C_0 and C_1 such that for evry $n\geq 2$*

$$C_0 \leq a_n \leq C_1 a_{n-1} \qquad (1)$$

and if

$$\sum_{n=1}^{\infty} M_n a_n^{-2+\delta/2} < \infty, \quad \sum_{k=n}^{\infty} a_k^{(\delta-3)/2} = 0(a_n^{(\delta-1)/2}),$$

then we have

$$\sum_{n=1}^{\infty} a_n^{-1+\delta/2} \| F_n(x)-\Phi(x) \|_p < \infty \qquad (1\leq p\leq\infty)$$

where $\| F_n(x)-\Phi(x) \|_p = (\int |F_n(x)-\Phi(x)|^p dx)^{1/p} \ (1\leq p\leq\infty)$

$$\| F_n(x)-\Phi(x) \|_\infty = \sup_x |F_n(x)-\Phi(x)|.$$

Corallary 2. *Let $\{X_n, n\geq 1\}$ be a sequence of independent and identically distributed random variables such that $EX_1=0$, $EX_1^2=1$, $EX_1^2\log X_1<\infty$. Suppose that* (1) *and if*

$$\sum_{n=1}^{\infty} M_n a_n^{-2} < \infty, \quad \sum_{k=n}^{\infty} a_n^{-3/2} = 0(a_k^{-1/2}),$$

then we have

$$\sum_{n=1}^{\infty} a_n^{-1} \| F_n(x) - \Phi(x) \|_p < \infty, \qquad (1 \leq p \leq \infty).$$

2. Proofs of Theorem 1 and Theorem 2.

Now we shall prove Theorem 1.

We have the following inequality from (10) of Rychlik and Szynal [9] p. 686 ;

$$sup\,|F_n(x) - \Phi(x/A_n)| \leq C[a_n P(|X_1| \geq \sqrt{a_n})$$

$$+\sqrt{a_n}\int_{|u|\leq\sqrt{a_n}} u dF(u) + a_n^{-1}\int_{|u|\sqrt{a_n}} |u|^3 dF(u) + M_n a_n^{-2}]. \qquad (2)$$

where $A_n^2 = \{\int_{|u|<\sqrt{a_n}} u^2 dF(u) - (\int_{|u|<\sqrt{a_n}} u dF(u))^2\}$.

It is easly seen that the first and second terms in the right hand side of the inequality (2) are dominated by

$$\int_{|u|\leq\sqrt{a_n}} u^2 dF(u). \qquad (3)$$

In order to estimate $|\Phi(x/A_n) - \Phi(x)|$, we use the following result of Erickson [3] p. 527 ;

$$|\Phi(ax) - \Phi(x)| \leq 4/5 \cdot |1 - a^{-1}| \qquad (a>0).$$

Therefore we have

$$|\Phi(x/A_n) - \Phi(x)| \leq |1 - A_n^2| \leq 2\int_{|u|\geq\sqrt{a_n}} u^2 dF(u). \qquad (4)$$

From (2) (3) and (4), we have the inequality of the Theorem 1. Thus the proof of Theorem 1 is complete.

Now we prove Theorem 2.

We define

$$X_{nj} = X_{nj}(x) = \begin{cases} X_j & \text{if } X_j \leq (1+|x|)\sqrt{a_n} \\ 0 & \text{otherwise} \end{cases}$$

for n, $j=1, 2, 3 \ldots$.

Let us put

$$S_{nk} = \sum_{j=1}^{k} X_{nj}, \qquad \mu_{nk} = kEX_{n1}, \qquad B_{nk}^2 = k\sigma^2 X_{n1}.$$

Further we define K_i $(i=1, 2, 3, 4)$ in the following way ;

$K_1 = \{k; |\mu_{nk}/\sqrt{k}| \geq |x|/2,\ k>a_n/2\}$,

$K_2 = \{k; |\mu_{nk}/\sqrt{k}| < |x|/2,\ k>a_n/2\}$,

$K_3=\{k;|\mu_{nk}/\sqrt{k}|\geq|x|/2,\ k<a_n/2\}$,

$K_4=\{k;|\mu_{nk}/\sqrt{k}|<|x|/2,\ k<a_n/2\}$.

For $|x|<1$, the required inequality follows from the theorem 1. Therefore it is sufficient to limit our considerations to the case $|x|\geq 1$, we have the inequality for all x.

$$|F_n(x)-\Phi(x)|\leq\sum_{k=1}^{\infty}p_k(n)|P(S_k<x\sqrt{k})-\Phi(x)|$$

$$\leq\sum_{k\in K_1}p_k(n)|\cdot|+\sum_{k\in K_2}p_k(n)|\cdot|+\sum_{k\in K_3}p_k(n)|\cdot|+$$

$$+\sum_{k\in K_4}p_k(n)|\cdot|,$$

$$=I_1+I_2+I_3+I_4,\ \text{say,} \tag{5}$$

where $|\cdot|=|P(S_k<x\sqrt{k})-\Phi(x)|$.

Now we treat I_1. Since $|\cdot|\leq 1$, we have

$$I_1=\sum_{k\in K_1}p_k(n)\leq 2\sum_{k\in K_1}p_k(n)|\mu_{nk}/x\sqrt{k}|\leq 2\sqrt{2}x^{-1}a_n^{-1/2}E|\mu_{Nn}|,$$

$$\leq 4\sqrt{2}(1+|x|)^{-1}a_n^{-1/2}a_n|\int_{|u|>(1+|x|)\sqrt{a_n}}udF(u)|,$$

$$\leq 4\sqrt{2}(1+|x|)^{-2}\int_{|u|>(1+|x|)\sqrt{a_n}}u^2dF(u). \tag{6}$$

Here we have used the property of K_1.

To estimate I_2, we need the following inequality for all x

$$I_2\leq\sum_{k\in K_2}p_k(n)|P(S_k<x\sqrt{k})-P(S_{nk}<x\sqrt{k})|+$$

$$+\sum_{k\in K_2}p_k(n)\left|P\left(\frac{S_{nk}-\mu_{nk}}{B_{nk}}<\frac{x\sqrt{k}-\mu_{nk}}{B_{nk}}\right)-\Phi\left(\frac{x\sqrt{k}-\mu_{nk}}{B_{nk}}\right)\right|+$$

$$+\sum_{k\in K_2}p_k(n)\left|\Phi\left(\frac{x\sqrt{k}-\mu_{nk}}{B_{nk}}\right)-\Phi\left(\frac{x\sqrt{k}}{B_{nk}}\right)\right|+$$

$$+\sum_{k\in K_2}p_k(n)\left|\Phi\left(\frac{x\sqrt{k_n}}{B_{nk}}\right)-\Phi(x)\right|,$$

$$=J_1+J_2+J_3+J_4,\ \text{say.} \tag{7}$$

Since $P(S_k<x\sqrt{k})\leq P(S_{nk}<x\sqrt{k})+P(S_k\neq S_{nk})$, we observe that

$$J_1=\sum_{k=1}^{\infty}p_k(n)P(S_k\neq S_{nk})\leq a_nP(|X_1|>(1+|x|)\sqrt{a_n})$$

$$\leq a_n\int_{|u|>(1+|x|)\sqrt{a_n}}dF(u),$$

$$\leq(1+|x|)^{-2}\int_{|u|>(1+|x|)\sqrt{a}}u^2dF(u). \tag{8}$$

To estimate J_2, using the inequality of Nagaev [6];

$$|P(S_k<x\sqrt{k})-\Phi(x)|\leqq C(1+|x|^3)^{-1}\sqrt{k}\,E|X_1|^3,$$

we have

$$J_2\leqq 4C\textstyle\sum_{k\in K_2}p_k(n)(1+|y_{nk}|^3)^{-1}B_{nk}^{-3}kE|X_1|^3,$$

whese $y_{nk}=(x\sqrt{k}-\mu_{nk})B_{nk}^{-1}$, However since

$$(1+|y_{nk}|^3)^{-1}\leqq B_{nk}^3k^{-3/2}|x-\mu_{nk}k^{-1/2}|^{-3}\leqq 8B_{nk}^3|x|^{-3}k^{-3/2},$$

we have

$$\begin{aligned}J_2&\leqq 32C\textstyle\sum_{k\in K_2}p_k(n)|x|^{-3}kE|X_{n1}|^3a_n^{-3/2},\\&\leqq 64C\textstyle\sum_{k\in K_2}p_k(n)|x|^{-3}ka_n^{-3/2}E|X_{n1}|^3,\\&\leqq 64\sqrt{2}\,C|x|^{-3}a_n^{-1/2}E|X_{n1}|^3.\end{aligned}\tag{9}$$

Here we have used the prorerty of K_2.

As to J_3, since exp $(u^2/2)>u^2/2$ for $u>0$, we have

$$\begin{aligned}J_3&\leqq\textstyle\sum_{k\in K_2}p_k(n)2/\sqrt{2\pi}\cdot\left|\displaystyle\int_{(x\sqrt{k}-\mu_{nk})B_{nk}^{-1}}^{x\sqrt{k}B_{nk}^{-1}}u^{-2}du\right|,\\&\leqq\sqrt{2}/\sqrt{\pi}\cdot\textstyle\sum_{k\in K_2}p_k(n)|-B_{nk}(x\sqrt{k})^{-1}+B_{nk}(x\sqrt{k}-\mu_{nk})^{-1}|,\\&\leqq 4/\sqrt{2\pi}\cdot\textstyle\sum_{k\in K_2}p_k(n)x^{-2}k^{-1/2}|\mu_{nk}|,\\&\leqq\sqrt{2}/\sqrt{\pi}\cdot|x|^{-1}a_n^{-1/2}\textstyle\sum_{k\in K_2}p_k(n)k|EX_{n1}|,\\&\leqq 2\sqrt{2}/\sqrt{\pi}\cdot(1+|x|)^{-2}\displaystyle\int_{|u|>(1+|x|)\sqrt{a_n}}u^2dF(u).\end{aligned}\tag{10}$$

Here we have used tha property of K_2.

Since $\exp(u^2/2)>u^3/8$, we have

$$\begin{aligned}J_4&\leqq 8/\sqrt{2\pi}\cdot\textstyle\sum_{k\in K_2}p_k(n)\left|\displaystyle\int_x^{x\sqrt{k}B_{nk}^{-1}}u^{-3}du\right|,\\&\leqq\sqrt{2}/\sqrt{\pi}\cdot\textstyle\sum_{k\in K_2}p_k(n)x^{-2}(k-B_{nk}^2)k^{-1}\\&\leqq 4\sqrt{2}/\sqrt{\pi}\cdot(1+|x|)^{-2}\displaystyle\int_{|u|>(1+|x|)\sqrt{a_n}}u^2dF(u).\end{aligned}\tag{11}$$

Hence from (7) (8) (9) (10) and (11), choosing a suitable constant, we have

$$\begin{aligned}I_2\leqq\ &C(1+|x|)^{-2}\int_{|u|>(1+|x|)\sqrt{a_n}}u^2dF(u)+\\&+C(1+|x|)^{-3}a_n^{-1/2}\int_{|u|\leqq(1+|x|)\sqrt{a_n}}|u|^3dF(u).\end{aligned}\tag{12}$$

Now we treat I_3. Since $|\cdot|\leqq 1$, we have

$$I_3\leqq 2\textstyle\sum_{k\in K_3}p_k(n)|\mu_{nk}/x\sqrt{k}|.$$

However

$$|EX_{n1}| = \left|\int_{|u|\geq(1+|x|)\sqrt{a_n}} u\,dF(u)\right| \leq (1+|x|)^{-1} a_n^{-1/2},$$

hence

$$|\mu_{nk}| \leq (1+|x|)^{-1} k a_n^{-1/2} \leq |x|^{-1} a_n^{-1/2} k.$$

Therefore we have

$$I_3 \leq 2\sum_{k\in K_3} p_k(n)|x|^{-2} a_n^{-1/2} k^{1/2} \leq \sqrt{2}\,|x|^{-2}\sum_{k\in K_3} p_k(n),$$

$$\leq 2\sqrt{2}\,(1+|x|)^{-2}\sum_{k;k\leq a_n/2} p_k(n),$$

$$\leq 4\sqrt{2}\,(1+|x|)^{-2} M_n a_n^{-2}. \tag{13}$$

At last we treat I_4. We need the inequality (7) replacing K_2 by K_4.

$$I_4 \leq J_1 + J_2 + J_3, \text{ say,} \tag{14}$$

where

$$J_3 = \sum_{k\in K_4} p_k(n)\,|\Phi((x\sqrt{k} - \mu_{nk})B_{nk}^{-1}) - \Phi(x)|.$$

From (12), we have

$$J_1 = \sum_{k\in K_4} p_k(n)\,|P(S_k < x\sqrt{k}) - P(S_{nk} < x\sqrt{k})|,$$

$$\leq (1+|x|)^{-2}\int_{|u|>(1+|x|)\sqrt{a_n}} u^2\,dF(u). \tag{15}$$

In order to estimate J_2, we use the following result of Agnew ([1] Theorem 2. 2) that if $F(x)$ and $G(x)$ and two distribution functions with zero mean and unit variance then

$$|F(x) - G(x)| \leq (1+x^2)^{-1} \qquad -\infty < x < \infty.$$

Then we have

$$J_2 \leq \sum_{k\in K_4} p_k(n)(1+|y_{nk}|^2)^{-1},$$

where $y_{nk} = (x\sqrt{k} - \mu_{nk})B_{nk}^{-1}$.

However since

$$(1+|y_{nk}|^2)^{-1} \leq B_{nk}^2 k^{-1}(x - \mu_{nk}k^{-1/2})^{-2} \leq 4B_{nk}^2 k^{-1} x^{-2} \leq 4x^{-2},$$

we have

$$J_2 \leq 8(1+|x|)^{-2}\sum_{k;k\leq a_n/2} p_k(n)$$

$$\leq 16(1+|x|)^{-2} M_n a_n^{-2}. \tag{16}$$

As to J_3, since $\exp(u^2/2) > u^3/8$, for $u > 1$, we have

$$J_3 \leq 8\sum_{k\in K_4} p_k(n)\left|\int_x^{(x\sqrt{k} - \mu_{nk})B_{nk}^{-1}} u^{-3}\,du\right|,$$

$$\leq 8\sum_{k\in K_4} p_k(n)\,|-B_{nk}^2(x\sqrt{k} - \mu_{nk})^{-2}2^{-1} + (2x^2)^{-1}|,$$

$$\leqq 24(1+|x|)^{-2} \sum_{k;k\leqq a_n/2}' p_k(n),$$

$$\leqq 48(1+|x|)^{-2} M_n a_n^{-2}. \tag{17}$$

Hence from (14) (15) (16) and (17), we have

$$I_4 \leqq C(1+|x|)^{-2}\int_{|u|>(1+|x|)\sqrt{a_n}} u^2 dF(u) + C(1+|X|)^{-2} M_n a_n^{-2}. \tag{20}$$

From (5) (6) (12) (13) and (20), choosing a suitable constant, we have the following inequality

$$|F_n(x)-\Phi(x)| \leqq C[(1+|x|)^{-3} a_n^{-1/2} \int_{|u|\leqq(1+|x|)\sqrt{a_n}} |u|^3 dF(u) +$$

$$+(1+|x|)^{-2}\int_{|u|>(1+|x|)\sqrt{a_n}} u^2 dF(u) + (1+|x|)^{-2} M_n a_n^{-2}]$$

$$= D_1 + D_2 + D_3, \text{ say.} \tag{21}$$

Being noted that

$$\int_0^{(1+|x|)\sqrt{a_n}} \int_{|u|>v} u^2 dF(u) dv = \int_{|u|<(1+|x|)\sqrt{a_n}} u^3 dF(u) +$$

$$+(1+|x|)\sqrt{a_n}\int_{|u|>(1+|x|)\sqrt{a_n}} u^2 dF(u),$$

we have the inequality in Theorem 2.

Thus the proof of Theorem 2 is complete.

3. Proofs of Theorem 3 and Corollaries.

In order to prove Theorem 3, we use Theorem 2. In the inequality (21), we have

$$D_1 = C(1+|x|)^{-3} a_n^{-1/2} \int_0^{(1+|x|)\sqrt{a_n}} u^3 dG(u),$$

$$\leqq C(1+|x|)^{-3} a_n^{-1/2} \int_0^{\sqrt{a_n}} u^3 dG(u) +$$

$$+ C(1+|x|)^{-2} \int_{\sqrt{a_n}}^{(1+|x|)\sqrt{a_n}} u^2 dG(n).$$

Sine $D_2 \leqq C(1+|x|)^{-2} \int_{(1+|x|)\sqrt{a_n}}^{\infty} u^2 dG(u),$

we have, noting $1+|x| \leqq (1+|x|)^2$

$$D_1 + D_2 \leqq C(1+|x|)^{-3} a_n^{-1/2} \int_0^{\sqrt{a_n}} u^3 dG(u) +$$

$$+ C(1+|x|)^{-2} \int_{\sqrt{a_n}}^{\infty} u^2 dG(u).$$

Therefore we have

$$D_1+D_2+D_3 \leqq C(1+|x|^2)^{-1}[a_n^{-1/2}\int_0^{\sqrt{a_n}} u^3 dG(u)+$$
$$+\int_{\sqrt{a_n}}^{\infty} u^2 dG(u)+M_n a_n^{-2}].$$

Thus we complete the proof of Theorem 3.

To show Corollary 1 and Corollary 2, we first observe that the series

$$\sum_{n=1}^{\infty} a_n^{-1+\delta/2} \sup_x |F_n(x)-\Phi(x)|, \quad \sum_{n=1}^{\infty} a_n^{-1} \sup_x |F_n(x)-\Phi(x)|$$

converge. However the convergences of these series are given by Theorem 2 and Theorem 3 of Rychlik and Szynal [9] in view of Theorem 1. For $1 \leqq p < \infty$, from Theorem 3, we have

$$\| F_n(x)-\Phi(x) \|_p \leqq C_n(\int |1+|x|^2|^{-p} dx)^{1/p}$$

where

$$C_n = C[a_n^{-1/2}\int_0^{\sqrt{a_n}} u^3 dG(u)+\int_{\sqrt{a_n}}^{\infty} u^2 dG(u)+M_n a_n^{-2}].$$

Hence the convergences of series

$$\sum_{n=1}^{\infty} a_n^{-1+\delta/2} \| F_n(x)-\Phi(x) \|_p, \quad \sum_{n=1}^{\infty} a_n^{-1} \| F_n(x)-\Phi(x) \|_p$$

are given by the proof of the case $p=\infty$.

REFERENCES

[1] Agnew, R. P. (1957). Estimates for global central limit theorems. Ann. Math. Stat. 28, 26-42.

[2] Bikyalis, A. (1966). Estimates for the remainder term in the central limit theorem. Litovsk. Matem. Sb. 6, 321-346.

[3] Erickson, R. V. (1974). L_1 bounds for asymptotic normality of m-dependent sums using Stein's technique. Ann. Prob. 2, 522-529.

[4] Galestyan, F. N. (1971). On the rate of convergence in the central limit theorem. Theo. Prob. Math, Stat. 18, 175-180.

[5] Heyde, C. C. (1975). A nonuniform bound on convergence to normality. Ann. Prob. 3. 5, 903-907.

[6] Nagaev, S. V. (1976). Some limit theorems for large deviations. Theo. Prob. Appl. 10, 214-235.

[7] Nakata, T. (1976). On the rate of convergence in mean central

limit theorem for martingale differences. Rep. Stat. Appl. Res. JUSE, 24, 3.

〔8〕 Nakata, T. (1977) A nonuniform bound on convergenc to normality for independent random variables. Bull. Fac. Lib. Arts. Chukyo Univ. 18, 19-24.

〔9〕 Rychlik, Z. and Szynal, D. (1974). On the convergence rates in the central limit theorem for the sums of a random number of i. i. d. r. v's. Bull. Acad. Poln. Sci., Ser. Sci. Math. Astronom. Phys., 22. 7, 683-690.

〔10〕 Rychlik, Z. and Szynal, D. (1975). Convergence rates in the central limit theorem for sums of a random number of independent r. v's. Theo. Prob. Appl. 20, 351-362.

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